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BENCHMARK COMPUTATION AND FINITE ELEMENT PERFORMANCE EVALUATION FOR A RHOMBIC PLATE BENDING PROBLEM

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The paper addresses the engineering benchmark computation of the bending rhombic plate. It shows the basic aspects and their influence on the performance of the finite element method.

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BENCHMARK COMPUTATION AND FINITE ELEMENT PERFORMANCE EVALUATION FOR A RHOMBIC PLATE BENDING PROBLEM

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1. INTRODUCTION

The performance evaluation of finite element methods has recently drawn a large attention. The question is strictly related to the development of a suitable set of benchmark problems upon which to verify and possibly to compare the accuracy of different finite elements. In here we refer to the proposed set of problems made by McNeal and Harder [1], the comparison studies of Robinson and Blackham [2], [3] and, for a comprehensive up-to-date review of the matter, the proceeding [4].

In this paper we consider the benchmark problem of a simply supported uniformly loaded rhombic plate. There are important questions related to the computation of such a problem. For example, which conclusions can be reasonably drawn from the results and how to interpret the rapid deterioration of the accuracy as the skewness becomes larger. Moreover, does the computation of the skew plate illustrate adequately the sensitivity of the finite elements to the skewness or is there some other more relevant effect to be considered? We shall show that the effect due to the skewness of the elements of the decomposition is negligible, the main effect being the singularity of the solution due to the presence of obtuse corners in the plate domain.

The design of benchmark problems should be made so as to be really representative for a well described class of problems and effects. Therefore, the selected problems need to be known in all their aspects that can influence the performance of the finite element solution. For example, it is well known that a special method or approach can be designed to perform very efficiently but only for a very narrow class of problems. Therefore attention should be placed also on the class of problems for which the test is characteristic.

Otherwise conclusion based on benchmark computations could be very misleading. There are, of course, many other questions. Among them: how to design "academic" benchmark problems isolating single effects which can then lead to useful conclusions for non-academic environments; how to assess robustness and reliability of the method; how to estimate the computational complexity of the method.

Obviously conclusions based on benchmark computation can never be completely objective. Nevertheless useful information can be drawn when based on the state of the art of both theoretical and experience field. The reliability has to be understood not only with respect to a particular mathematical model but, in addition, also the model has to be considered. For example, how accurate is the solution of the Kirchhoff model of a rhombic plate compared with the solution of three dimensional linear elasticity problem.

Throughout the paper we will address the above questions, focusing mainly on which type of conclusions can be inferred from the results of computation with different finite elements of a <u>simply supported uniformly loaded rhombic plate</u>. The following aspects will be especially analyzed:

1. Effect of the skewness of the element of the decomposition.
2. Effect of the singularity of the exact solution.
3. How to improve the performance of the finite element solution when the skew angle of the plate become small.
4. Relation between accuracy and computational complexity for various finite element methods on a given class of problems.
5. Class of problems the benchmark skew plate is representative of.

The outline of the paper is the following. In Section 2 we introduce the model problem and characterize, in a suitable mathematical way, the properties of the exact solution of the problem. Section 3 is devoted to the description of the finite elements used for computations, together with some abstract convergence results. In Section 4 we present the numerical results. First we consider the case of uniform decomposition, then we consider an appropriate non-uniform decomposition allowing to considerably reduce the magnitude of the error. The study of the relation between accuracy and cost of computation ends Section 4. Finally, in Section 5 the conclusions are shown.

2. THE MODEL PROBLEM

We are interested in the Kirchhoff model for the simply supported plate with parallelogram shape. Let us denote by Ω the domain of the (x_1,x_2)-plane occupied by the plate, let $\Gamma = \bigcup_{i=1}^{4}$ be its boundary, A_i, $i = 1,4$, its vertices and $\bar{\Omega} = \Omega \cup \Gamma$. Let α denote the skew angle of the plate.

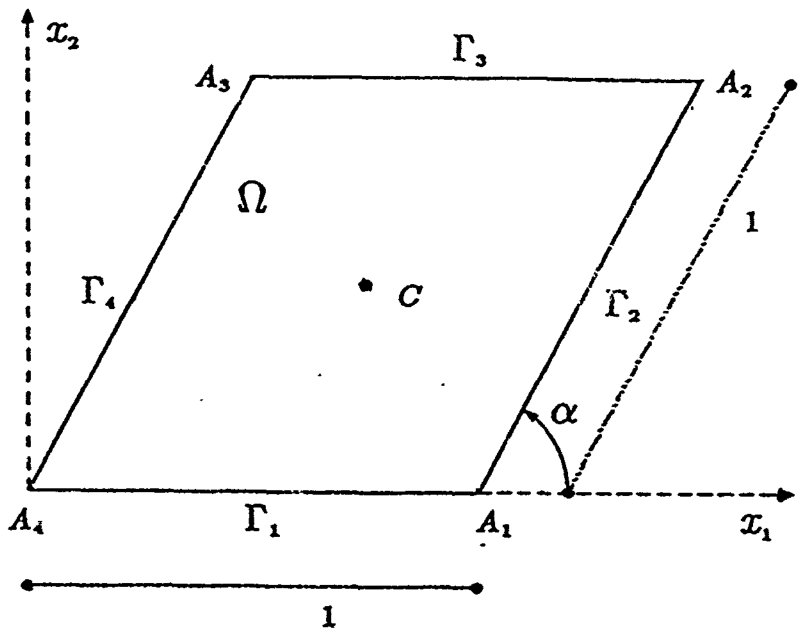

Fig. 2.1. The domain of the plate.

We assume the plate loaded by a transversal load $g(x_1,x_2)$. We denote by $H^k(\Omega)$ the standard Sobolev space of the functions with square-integrable derivatives up to the order k. We will also use the Sobolev spaces with fractional derivatives defined in the usual way by interpolation techniques (e.g. the K-method; see [5] for details). We also mention the Besov spaces $B^k_{2,\infty}(\Omega)$ which are very close the the spaces $H^k(\Omega)$, namely

2.1 $$H^k(\Omega) \subset B^k_{2,\infty}(\Omega) \subset H^{k-\varepsilon}(\Omega) \qquad \varepsilon > 0$$

(see [5] for more details). Further we denote

2.2 $$\overset{0}{H}{}^2(\Omega) = \{u \in H^2(\Omega) : u|_\Gamma = 0\}.$$

The exact solution u_0 of our problem can now be defined as the minimizer of the quadratical functional $F(u)$ over $\overset{0}{H}{}^2(\Omega)$, where

2.3 $$F(u) = B(u,u) - Q(u),$$

2.4 $$B(u,u) = \frac{1}{2} D\left\{\int_\Omega \left[\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2\right)\right] dx_1 dx_2\right\}$$

2.5 $$Q(u) = \int_\Omega g(x_1,x_2)\, u \, dx_1 dx_2$$

2.6 $$D = \frac{Et^3}{12(1-\nu^2)},$$

E is the Young's modulus, t the thickness of the plate and ν the Poisson's ratio.

The expression $B(u,u)$ has the meaning of the (strain) energy of the plate,

2.7 $$\gamma_1 \|u\|_{H^2(\Omega)} \geq \|u\|_E = B(u,u)^{1/2} \geq \gamma_2 \|u\|_{H^2(\Omega)}, \qquad \gamma_i > 0,$$

and has all the properties of a norm. Later we will measure the error

2.8 $$e = u_0 - u_{FE}$$

between the exact solution u_0 and the finite element solution u_{FE} in the (energy) norm defined by 2.7. It is easy to see that this measure is equivalent to the error in the moments measured in $L_2(\Omega)$, i.e. in the least square way over Ω.

Remark _.1. The formulation 2.3-2.5 has proper meaning for a general domain Ω.

It follows easily by 2.3 and 2.7 that the solution u_0 exists for any given load $g(x_1,x_2) \in H^0(\Omega) = L_2(\Omega)$ (i.e. space of the square integrable functions) and that it is unique.

Remark 2.2. The solution exists for a large class of loads. For example, a concentrated load ($g(x_1,x_2)$ = Dirac's function) is allowed due to the inclusion $H^2(\Omega) \hookrightarrow C^0(\Omega)$.

In the following we will concentrate on the case of loads given by analytic functions on $\bar{\Omega}$. A representative of this class is $g(x_1,x_2) = 1$, i.e. uniformly distributed load. (This example will be considered in the benchmark problem). In this case the solution u_0 is an analytic function on $\bar{\Omega} - \bigcup_{i=1}^{4} A_i$, i.e. u_0 is not analytic at the vertices A_i but is analytic at all the rest of the boundary. In the neighborhood of the vertex A_1 (and A_3) the solution has the form

$$2.9 \qquad u_0 = cr^{\frac{\pi}{\pi-\alpha}} \sin \frac{\pi}{\pi-\alpha}\theta + \text{smoother terms},$$

where α $(0 < \alpha < \pi/2)$ is the angle indicated in Fig. 2.2a. In the neighborhood of the vertex A_4 (and A_2) we have

2.10 $$u_0 = cr^{\frac{\pi}{\alpha}} \sin \frac{\pi}{\alpha} \theta + \text{smoother terms.}$$

With r,θ we denote the polar coordinates with the origin in A_1 (resp. A_2) as shown in Fig. 2.2a,b.

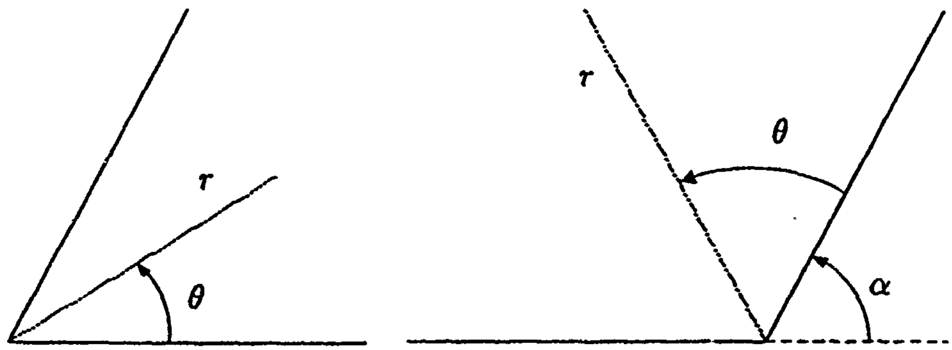

Figure 2.2a,b. Polar coordinates around the corners.

The solution u_0 has a strongest singular behaviour in the neighborhoods of the vertices A_1 and A_3. We see that the singularity becomes stronger as $\alpha \to 0$ and, for any α, $u_0 \notin H^3(\Omega)$. In fact, we can show that, for $\frac{\pi}{2} > \alpha > 0$

2.11 $$u_0 \in H^{\gamma-\varepsilon}(\Omega), \qquad \forall \varepsilon > 0,$$

2.12 $$\gamma = 2 + \frac{\alpha}{\pi-\alpha}.$$

Moreover

2.13 $$u_0 \notin H^{\gamma}(\Omega) \text{ but } u_0 \in B^{\gamma}_{2,\infty}(\Omega).$$

<u>Remark 2.3</u>. The regularity of the exact solution of the problem plays a crucial role in the performance of the finite element method. In [6] we have

characterized the smoothness of the solution in the framework of countably normed spaces and used those results in relation with the analysis of the performance of the h-p version of the finite element method (see [7,8], [9], [10]).

Remark 2.4. The singularity of the solution of the simply supported Kirchhoff plate caused by the corner of the domain leads to some paradoxical properties of the solution. Consider, for example, the problem of a simply supported plate with Poisson's ratio $\nu = 0$, with the shape of a regular n-sided polygon inscribed into a circle of radius R, uniformly loaded with a load q. Let u_n be the solution. Further, let u_c be the solution of the analogous problem for the circular plate of radius R. The solution u_n is defined by 2.3-2.5 and it is uniquely determined for every integer $n, n = 3,4,\ldots$ and $u_n \to u_\infty$ as $n \to \infty$, but, and this is the paradox, $u_\infty \neq u_c$. At the center C of the plate we have

2.14 $$u_\infty(0,0) = \lim_{n\to\infty} u_n(0,0) = -\frac{3}{64} R^4 \frac{q}{I},$$

2.15 $$u_c(0,0) = -\frac{5}{64} R^4 \frac{q}{I},$$

where I is the momentum of inertia. This means a difference of more than 40%!. The implications of this result (referred to as Babuška paradox) have been addressed in various papers (see e.g. [11], [12], [13], [14]).

3. THE FINITE ELEMENT APPROXIMATION

We are interested to solve numerically the problem defined by 2.3-2.5 using a finite element approximation. To this end we define the finite dimensional space S of the finite element solutions and then we minimize the functional $F(u)$ in 2.3 over the space S. The core of the finite element method, in its simplest form, is essentially the construction of a suitable finite element space S. First a triangulation (or other partition) is established over the set $\bar{\Omega}$ and then the space S is constructed. The quality of the finite element solution is determined by the properties of S.

We will consider in the following three different finite elements for plate bending problems:

a) Argyris element (ARGY)

b) reduced Hsieh-Clough-Tocher element (HCTR)

c) dual hybrid element (HYBR).

It is well known that a quite large number of elements for plate is described in the scientific literature and many of them are implemented in the finite element codes. A rough but effective classification can be made dividing the elements in two categories: conforming and non-conforming, depending on whether or not $S \subset {}^0H^2(\Omega)$. We have restricted curselves to the above mentioned elements, all of them conforming, although similar analysis as we present could be carried out for other elements as well.

Let us briefly describe the finite elements we have used, together with their basic properties.

The Argyris element

The conformity of a plate element can be obtained in different ways. In the Argyris element the C^1-condition (i.e. continuity of the functions of S and of their first derivatives), which implies $S \subset {}^0H^2(\Omega)$ is satisfied through the use of a complete space of degree 5 (see [15], [16]). This corresponds to 21 degrees of freedom per triangle, as shown in Fig. 3.1. In particular the value of the displacement, together with its first and second derivatives, is imposed at the vertices, the normal derivative is prescribed at the midpoint of each side. We recall that the number of d.o.f. is very close of the optimal one (18, see [17]) needed to insure the conformity when (only) a polynomial space is used.

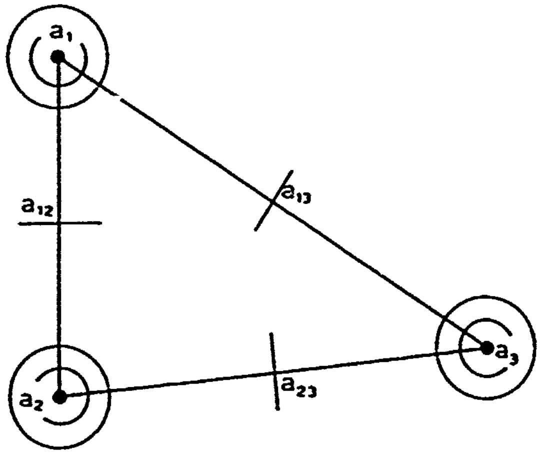

Fig. 3.1. The Argyris element (ARGY).

The reduced Hsieh-Clough-Tocher element

The original Hsieh-Clough-Tocher element is a composite one: the basic element is subdivided into three triangles and then on each subtriangle a complete cubic polynomial is defined. After imposing the C^1-continuity at the vertices and across the sides of the subtriangles only 12 of the initial

30 d.o.f. remain: displacement and its first derivatives at the vertices, normal derivative at the midpoint of each side. The reduced element (see Fig. 3.2) is obtained eliminating the degrees of freedom normal derivatives. This correspond to require the cubic polynomials on each subtriangle to have linear normal derivatives on the sides. The dimension of the space associated to the element, 9, is optimal in the sense that this is the lower possible number of d.o.f. required for a conforming element. We mention that this element is used in several commercial codes. For more details we refer to [18], [19], [20].

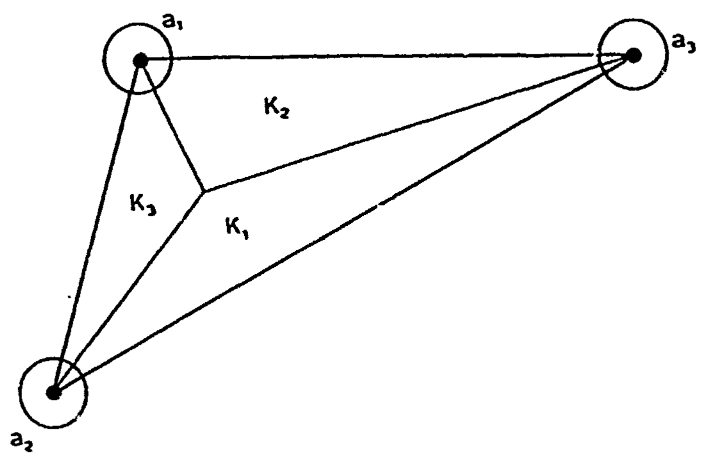

Fig. 3.2. The reduced Hsieh-Clough-Tocher element (HCTR).

The dual hybrid element

The basic idea for the so-called hybrid finite elements was first suggested by Pian and Tong [21]. We refer to [22] for a clear exposition of the features of this approach for solving linear solid mechanics problems. The element we have used has been extensively analyzed, both from theoretical and numerical standpoint, by Brezzi and Marini (see [23], [24], [25]). The main aspect of the finite element is the particular choice of the displacement approximation space, formed by cubic polynomials only defined at the boundary of the triangle. Obviously, the functions are in some way extended to the whole triangle, but the computations requires only the values at the interelement boundaries The degrees of freedom are shown in Fig. 3.3 and consists of values of displacement and its first derivatives at the vertices. The dimension of the approximation space is 9.

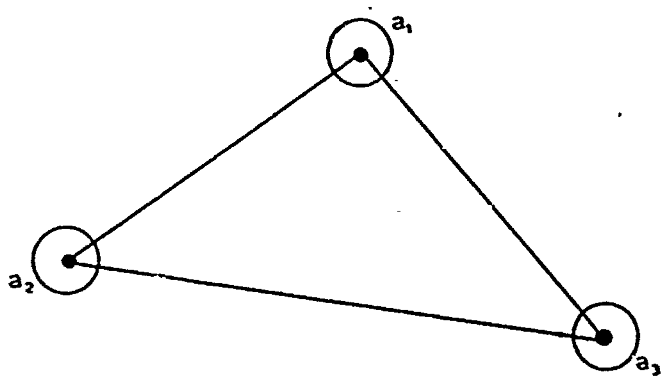

Fig. 3.3. The dual hybrid element (HYBR).

Let us now recall the approximation properties of the elements we have:

THEOREM 3.1. Let the triangulation of $\bar{\Omega}$ be quasi-uniform(satisfying the minimal angle condition), the solution $u_0 \in H^k(\Omega) \cap {}^0H^2(\Omega)$, $k > 2$ (integer or fractional). Thenthe following estimate holds:

3.1 $$\|u_0 - u_{FE}\|_E \leq Ch^\mu \|u_0\|_{H^k(\Omega)}$$

with

3.2 $$\mu = \min(k-2,4) \text{ for ARGY}$$

3.3 $$\mu = \min(k-2,1) \text{ for HCTR, HYBR,}$$

where h is the maximum size of the elements of the decomposition, C is a constant depending on the aspect ratio, the skewness of the elements and the type of the finite element.

Theorem 3.1 is related to the standard h-version of the finite element method. As the number N of degrees of freedom is (asymptotically) of order h^{-2} we can re-write relation 3.1 in the form

3.4 $$\| u_0 - u_{FE} \|_E \leq CN^{-\frac{1}{2}k} \| u_0 \|_{H^K(\Omega)} .$$

Theorem 3.1 is formulated in a general way. For the problem of uniformly loaded simply supported skew plate we can state the following:

THEOREM 3.2. Let the triangulation of $\bar{\Omega}$ be quasi-uniform (satisfying the minimal angle condition) and the load be uniform on $\bar{\Omega}$. Then there exist two constant C_1 and C_2 such that the following estimate holds:

3.5 $$C_1 N^{-\frac{1}{2}\frac{\alpha}{\pi-\alpha}} \leq \| u_0 - u_{FE} \|_E \leq C_2 N^{-\frac{1}{2}\frac{\alpha}{\pi-\alpha}} ,$$

where the constants C_1 and C_2 depend on the aspect ration, the skewness of the elements, the finite element itself (ARGY, HCTR or HYBR), but are independent of N (i.e. the rate of convergence is the same for all three elements).

Theorem 3.2 shows that the rate of convergence, and therefore the accuracy, deteriorates while the skew angle α decreases. This behaviour is almost independent of aspect ratio and skewness of the elements as these influence only the constants C_1 and C_2. The theorem holds not only for our choice of elements but it is more general, the only exception being the case when special elements with the shape functions of the form 2.9 are used (this is, for example, the reason of the performance of the element ELFIN in the comparative study made by Robinson [26]).

Theorem 3.2 has only asymptotic character when the constants are not specified. Obviously, in practice, these constants are important and benchmark computations may characterize them well for particular classes of problems.

So far we have addressed only the case of quasi-uniform mesh. If the mesh is properly refined then we have the following:

THEOREM 3.3. Let the load acting on the plate be uniform on $\bar{\Omega}$ and the mesh decomposition properly selected. Then there exist constant C_1 such that the following estimate holds:

$$\|u_0 - u_{FE}\|_E \leq C_1 N^{-\frac{1}{2}\eta}, \tag{3.6}$$

with $\eta = 4$ for ARGY, $\eta = 1$ for HCTR and HYBR. Further for any mesh $C_2 N^{-\frac{1}{2}\eta} \leq \|u_0 - u_{FE}\|_E$ where $C_2 > 0$ depends only on skewnees of the elements. □

Theorem 3.3 shows that the performance of ARGY is especially good when proper design of the mesh is made. We will show that this choice will permit the achievement of a higher accuracy.

Theorems 3.1, 3.2 and 3.3 follow from the standard mathematical theory of finite elements (see e.g. [19], [27]).

So far we have discussed only the performance with respect to the energy norm measure of the error. Analogous behaviour is expected for other norms of the accuracy. We will partially address questions of this type in the following section.

4. THE BENCHMARK PROBLEM COMPUTATION

In this section we compare the performance of the finite elements described in Section 3. We will analyze energy and displacement errors with respect to the numbers of degrees of freedom and the cost of computational work. Quite often in the literature only the relation between error and degrees of freedom is shown. Although this is a very important characterization, the most important information is the relation between accuracy and computational cost. The latter depends, of course, on different factors (programming technique, solver algorithm, etc.) which make it not completely well defined while the first characterization has a completely precise meaning. Nevertheless, using standard level of programming and suitable description of the machine cost very reliable informations can be obtained.

4.1. THE UNIFORM MESH DECOMPOSITION

We first consider the case of uniform mesh. In Fig. 4.1 an example of such a mesh is shown.

We have solved the skew plate problem for four different values of the skew angle α: 80°, 60°, 40°, 30°.

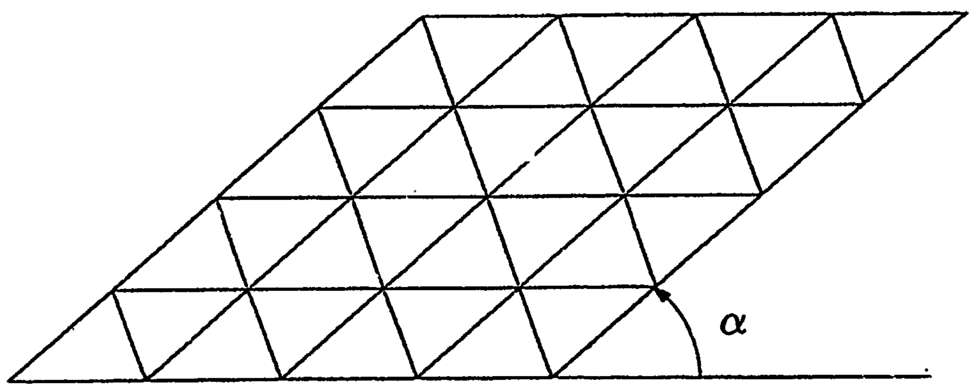

Fig. 4.1. A uniform mesh with triangular elements.

The physical data of the plate are the following:

side length	1.0 in
Young's modulus	$3. \times 10^7$ lb/in^2
Poisson's ratio	0.3
thickness	0.1 in
load	1. lb/in^2

First we show the relation between the energy norm of the error and the number of degrees of freedom. Let E_{EX} denote the exact energy of the plate, E_{FE} the energy of the finite element solution. The relative energy norm $\|e\|_{ER}$ of the error $e = u_0 - u_{FE}$ can be expressed (using basic properties of the finite element solution) in the following way

$$\|e\|_{ER} = \frac{\|e\|_E}{\|u_0\|_E} \% = \left(\frac{E_{EX}-E_{FE}}{E_{EX}}\right)^{\frac{1}{2}} 100. \tag{4.1}$$

Figures 4.2a-d show the results for different values of α when a mesh of the type shown in Fig. 4.1 is used. In each figure results for ARGY, HCTR and HYBR elements are given. Both the scales are of logarithmic type. In all the cases the ARGY element give better performances. This is expected when the singularity is still weak (i.e. $\alpha = 80°$), but it happens for each value of α which supplies the information about the constants C_1 and C_2 (see Theorem 3.2). Within each test the order of convergence is nearly the same

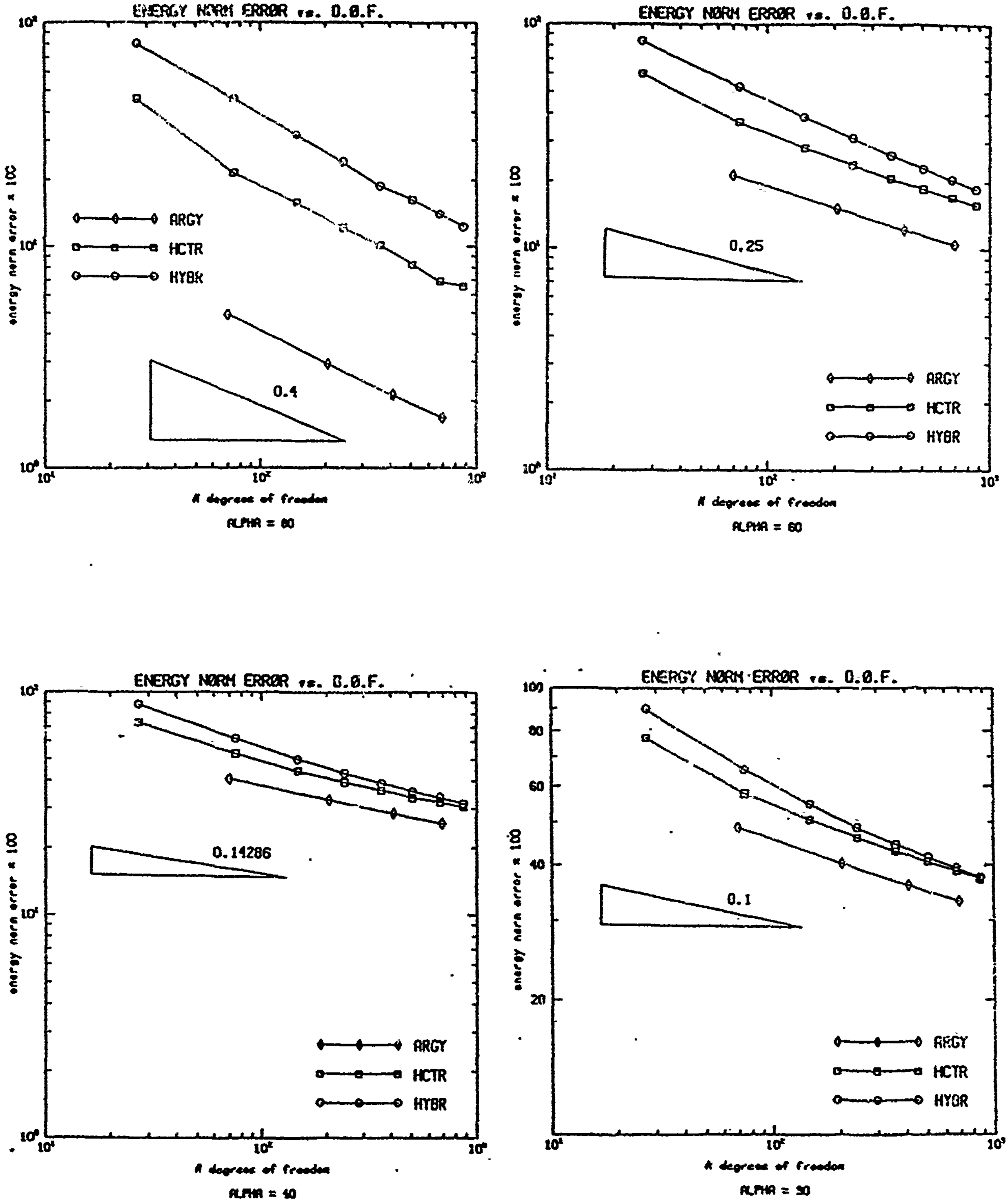

Fig.4.2a-d. Energy norm of the error against number of degrees of freedom for uniform mesh. The slope of the triangle denotes, in each figure, the theoretical rate of convergence.

for the three different methods. In the figures the theoretical order is shown. As the error is (see 3.5) of order

$$N^{-\frac{1}{2}\frac{\alpha}{\pi-\alpha}} \tag{4.2}$$

the slope of the lines is expected to be close to the value $-\frac{1}{2}\frac{\alpha}{\pi-\alpha}$. A quite good agreement between theoretical and computational order of convergence can be seen. We note, as expected, the rapid increasing of the error when α becomes smaller. In particular in Fig. 4.2a,b,c the same scales have been used to emphasize the deterioration of the accuracy while α assumes the values 80, 60, 40. Due to the magnitude of the error a different scale is used in Fig. 4.2d ($\alpha = 30°$).

Let us now consider the displacement of the plate. Let $u_{MO}(O)$ denote the value at the center of the plate computed by Morley ([28], [29]) using series expansion, $u_{FE}(C)$ the finite element solution. The relative displacement error is simply defined as

$$D\% = \left(\frac{u_{MO}(C) - u_{FE}(C)}{u_{MO}(C)}\right) \cdot 100. \tag{4.3}$$

Figures 4.3a-d give the error D% against number of degrees of freedom. The behaviour of the displacement error is roughly the same as the error in the energy ~~norm~~. This is not surprising due to the relation linking energy and displacement (i.e. the error in the energy is the average error in displacement).

Let us now explain the reason of the deterioration of accuracy as α becomes smaller. Obviously when α decreases the skewness of the elements increases and, at the same time, the results worsen. This fact sometimes

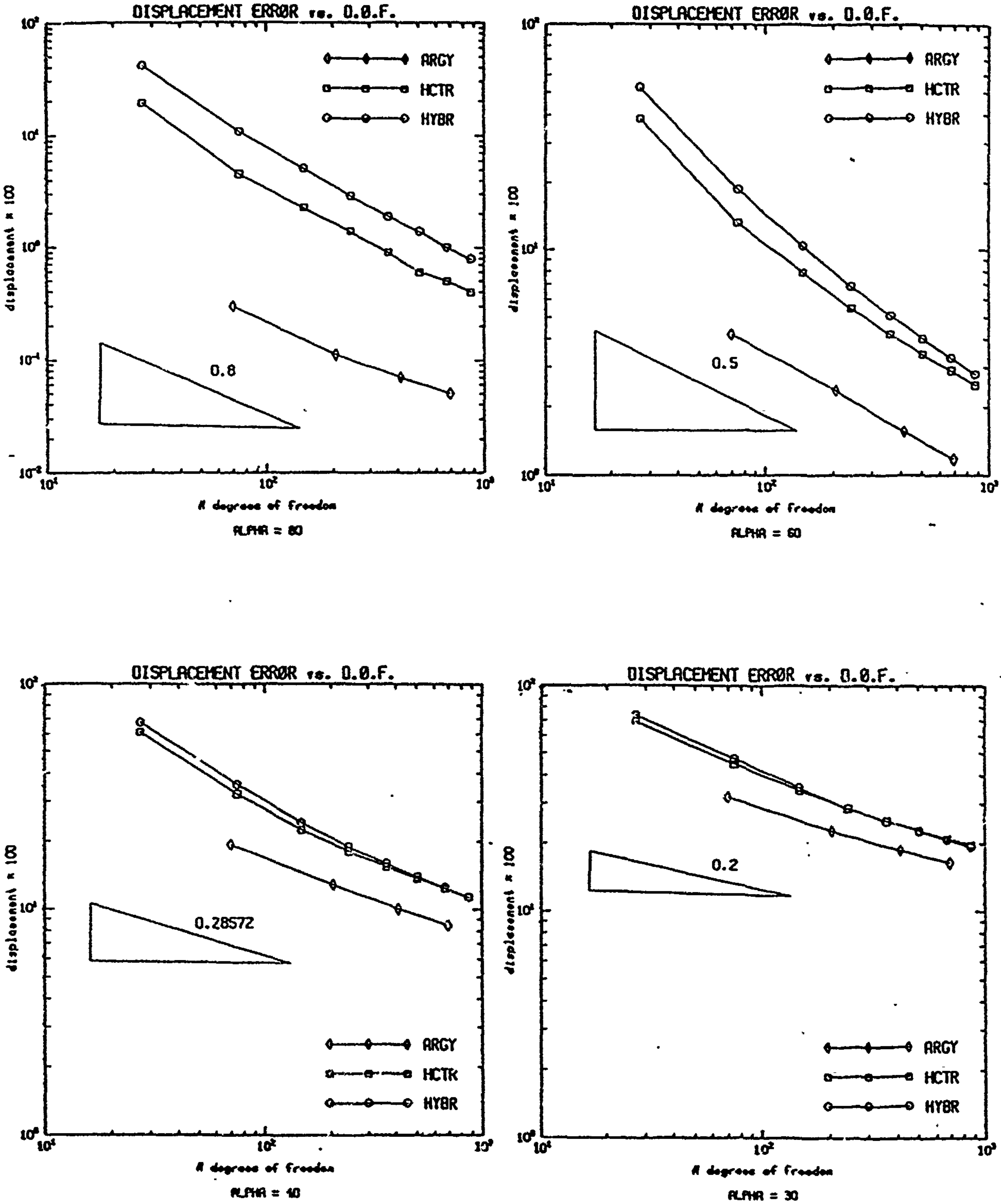

Fig. 4.3a-d. Relative displacement error at the center of the plate against number of degrees of freedom for uniform mesh. The slope of the triangle denotes, in each figure, the theoretical rate of convergence.

seems to lead to the conclusion (see [26]) that the effect of the skewness of the elements is the reason of worsening. The major factor is the change of the smoothness of the exact solution in dependance of α and not the skewness of the elements. To illustrate this fact let us compute the error when meshes of type shown in Fig. 4.4 b and d (in Fig. 4.4 a and c the usual meshes are shown). Of course, the skewness of meshes 4.4 b and d is larger than the one of 4.4 a and c, respectively. Figures 4.5a-d compare the measure in the energy norm of the error for both normal and 'skewed' meshes. The continuous lines refer to the normal mesh, the dashed lines to the 'skewed' mesh. Figures 4.6a-d show analogous results for the displacement error.

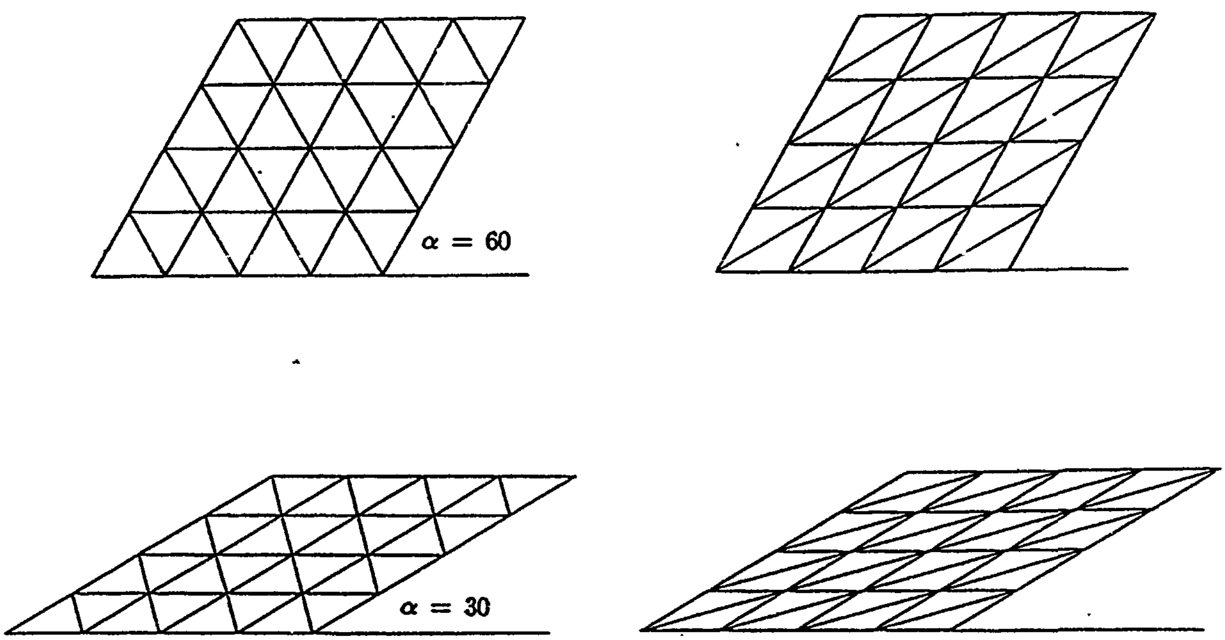

Fig. 4.4a-d. Examples of mesh used to evaluate the sensitiveness of the numerical solution to the skewness of the elements of the decomposition. Left: 'natural' mesh; right: 'skewed' mesh.

We see that the singularity of the solution and not the skewness of the elements is the main reason for the change of performance of the method. Only a relatively slight contribution is given by the 'skewed' mesh negligible compared with the influence of the singularity.

Remark 4.1. Of course the form of the triangle, more precisely its maximal angle (see [30]) has essential influence on convergence (i.e. when it is violated convergence does not occur at all). Nevertheless, in our case the maximal angle of any triangle is far enough from π and thus its effect is negligible compared with the smoothness effect.

4.2. The non-uniform mesh decomposition

The presence of singular behaviour at the obtuse corners of the plate makes the moments unbounded. The rate of convergence (see 4.2) is so small that uniform mesh gives completely unacceptable results. In example, for $\alpha = 30°$, extrapolating the error behaviour shown in Fig. 4.3d, we see that a number of degrees of freedom $N > 10^6$(!!) is needed to achieve a 5% accuracy for the displacement at the center of the plate. If we properly refine mesh then Theorem 3.3 applies and much better results can be achieved, especially for the ARGY element. The optimal meshes need to be graded in the place where singular solution occurs. Figures 4.7a-d show a sequences of refined meshes used in the computation. Figures 4,8a-d show the error in the energy norm against the number of degrees of freedom for angles $\alpha = 80°, 60°, 40°, 30°$. The performance of the ARGY element with the non-uniform meshes of Fig. 4.7 is reported, together with the results related to uniform meshes. Also the slope related to the theoretical order given in Theorem 3.3 is shown in the figures. The improvement of the accuracy is really effective, especially when α becomes smaller.

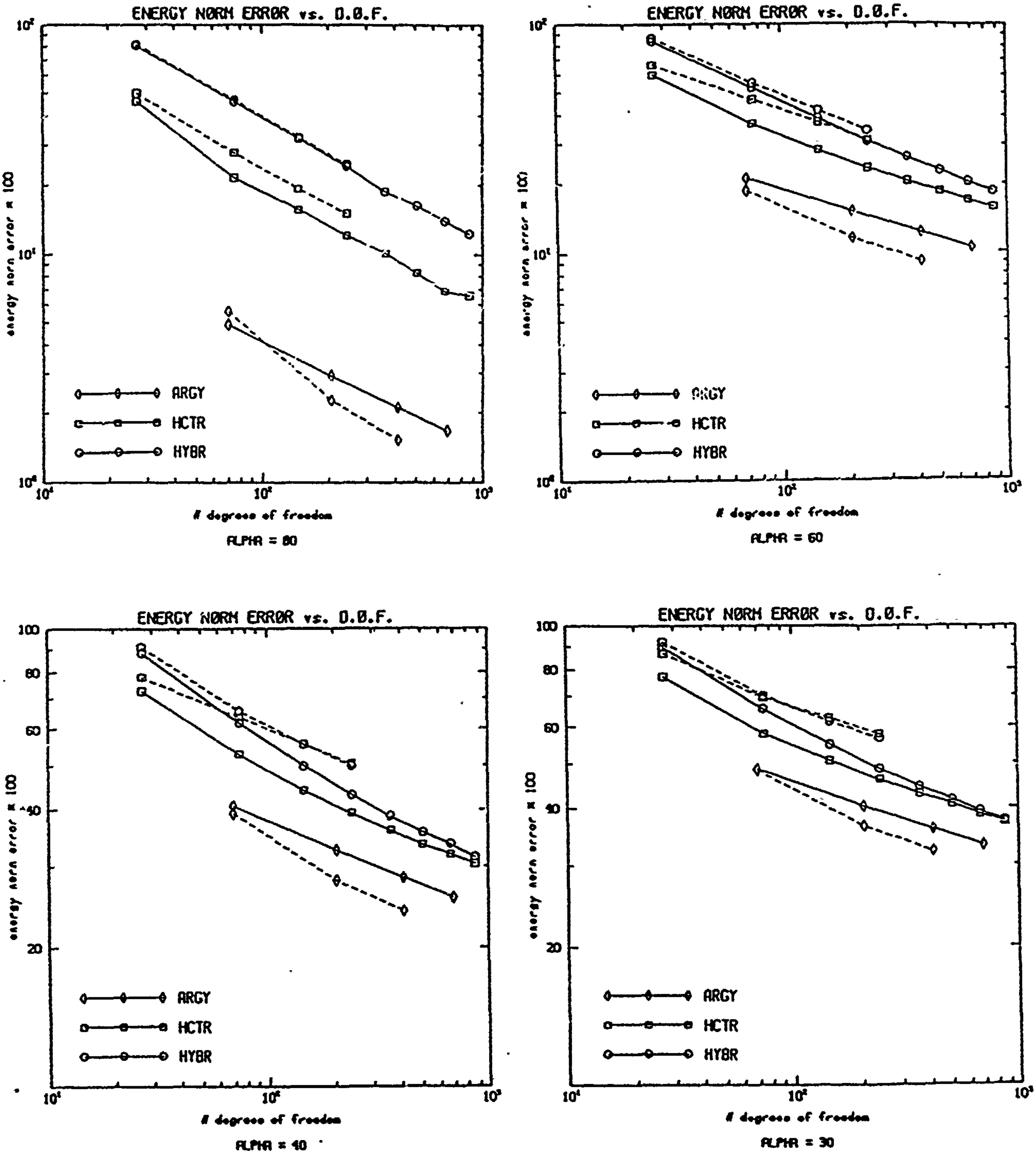

Fig. 4.5a-d. Energy norm of the error against number of degrees of freedom for uniform mesh: with 'natural' mesh (continuous line) and 'skewed' mesh (dashed line).

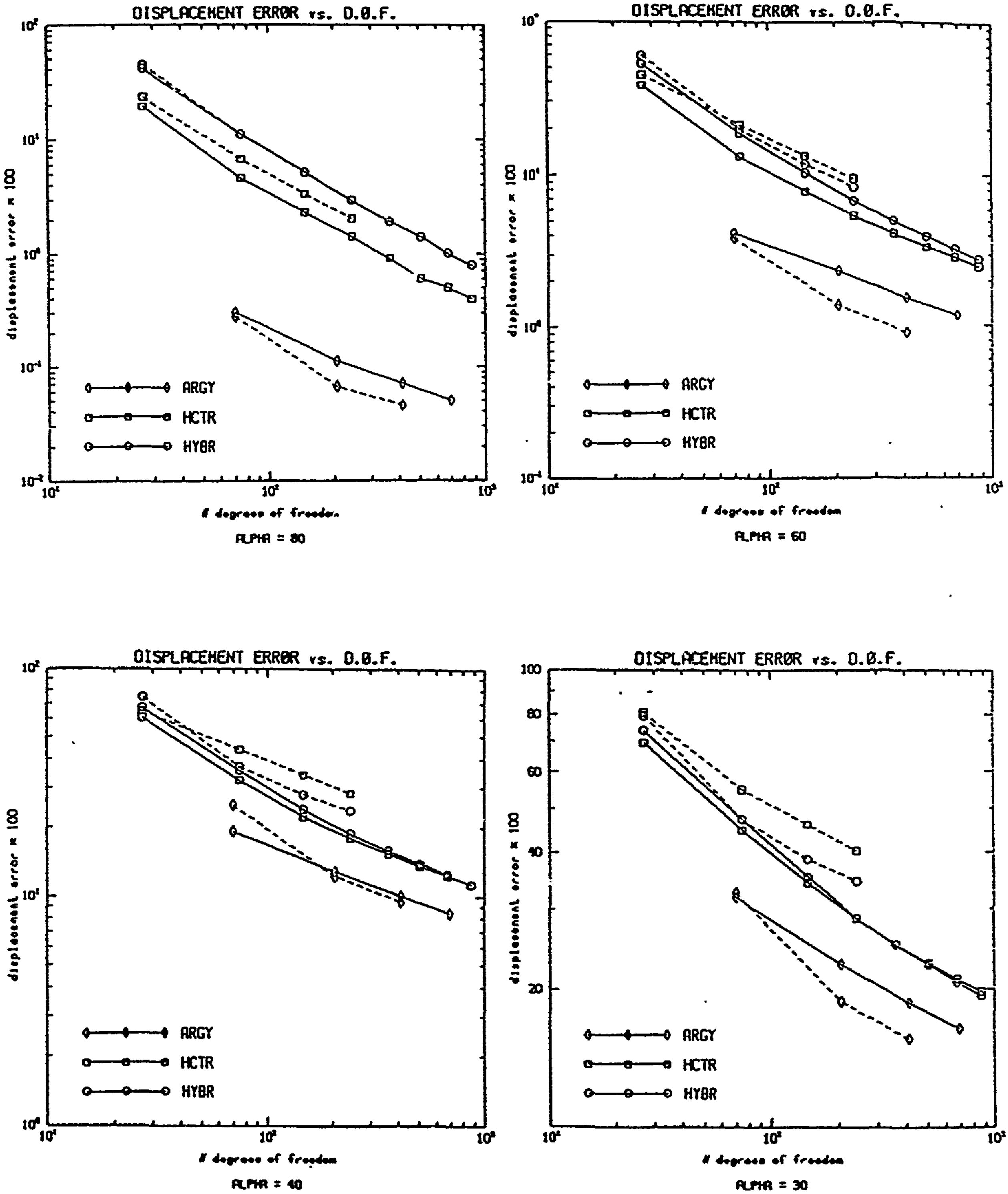

Fig. 4.6a-d. Relative displacement error at the center of the plate against number of degrees of freedom for uniform mesh: with 'natural' mesh (continuous line) and 'skewed' mesh (dashed line).

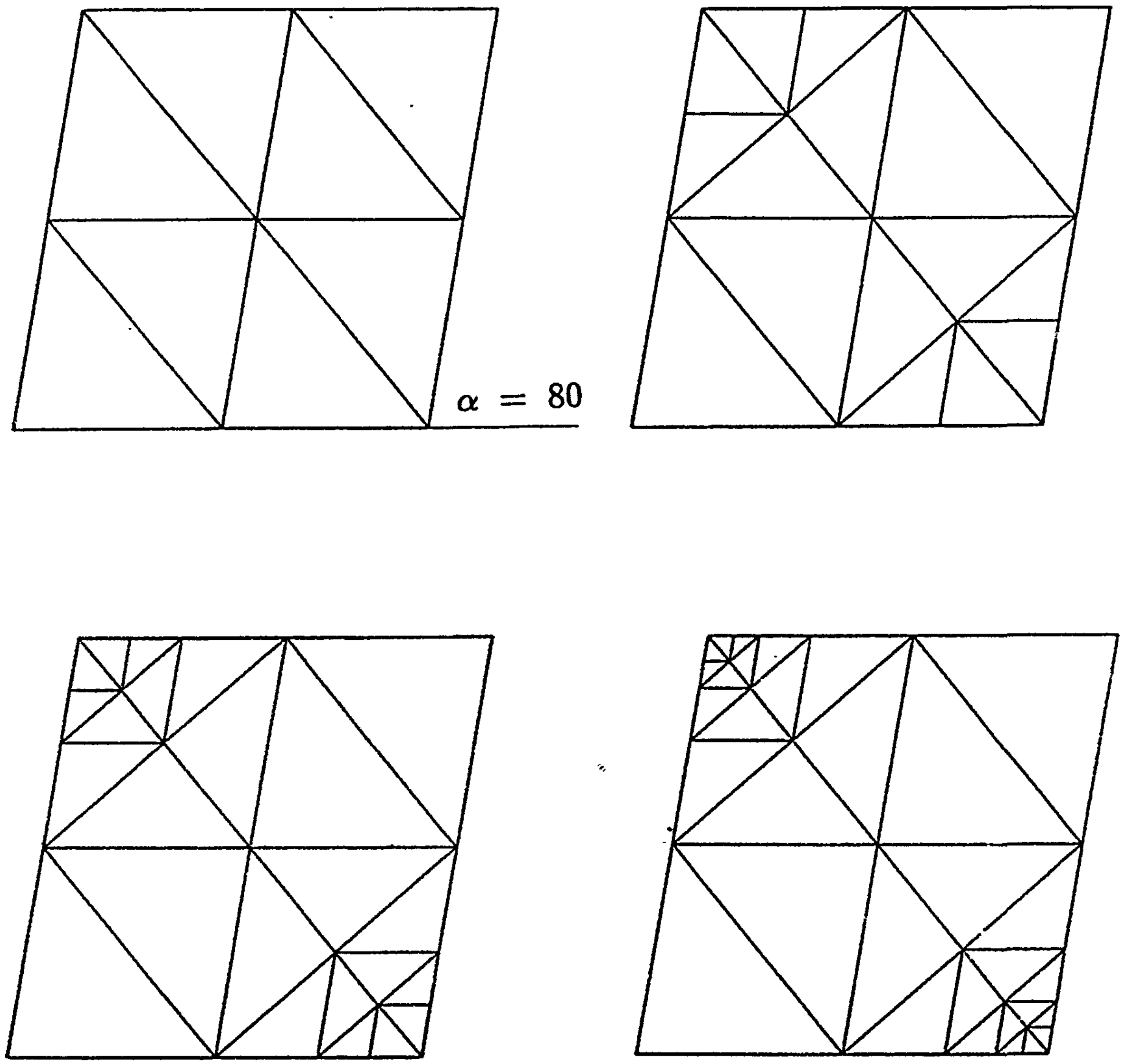

Fig.4.7a-d. A sequence of non-uniform decompositions with refinement around the obtuse corners.

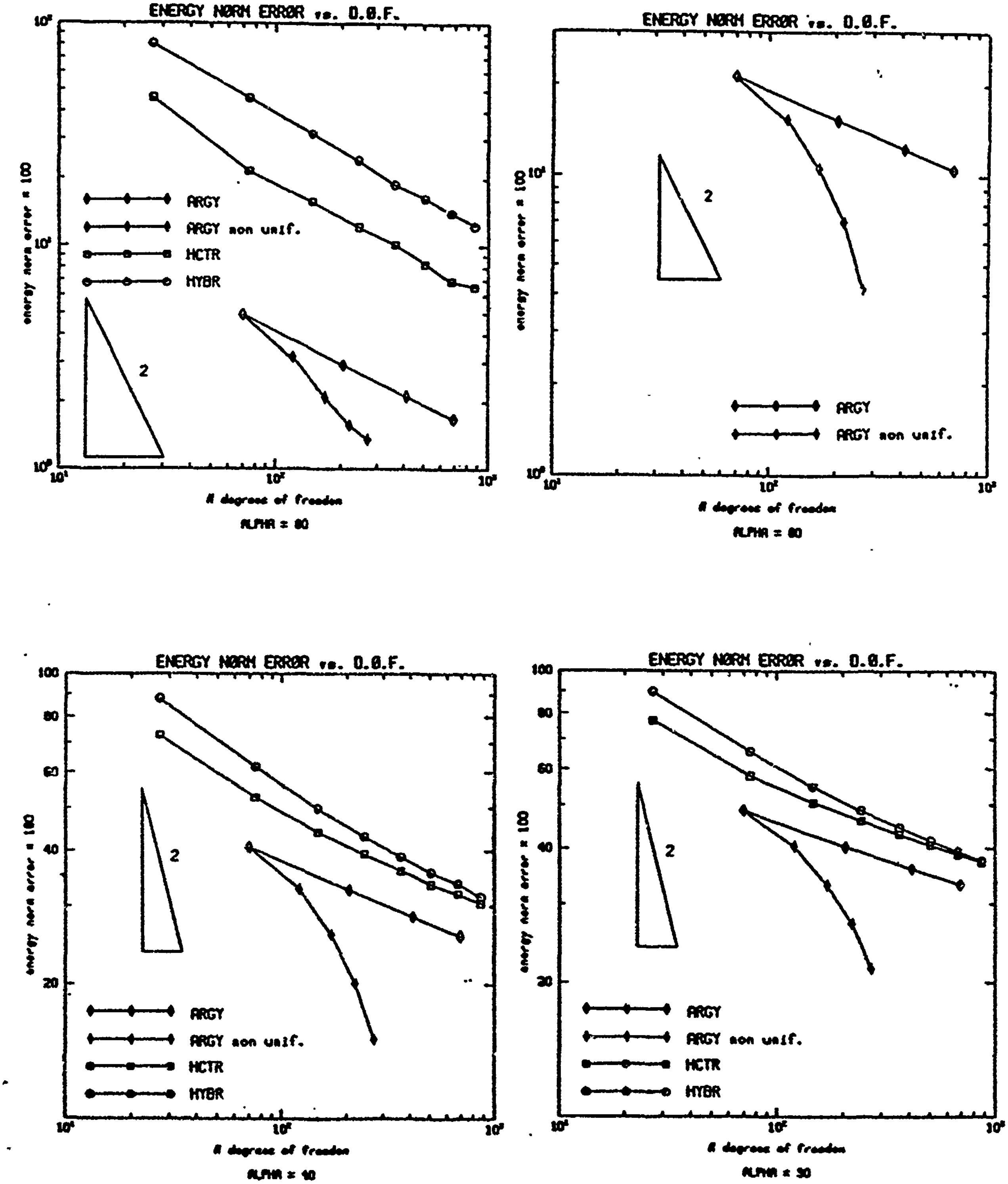

Fig. 4.8a-d. Energy norm of the error against number of degrees of freedom for uniform and non-uniform mesh.

The mesh shown in Fig. 4.7 are geometric meshes with a factor ½. This type of decomposition is effective as far as the major contribution to the error appears in the smallest element, due to the singularity of the solution. If the singularity is not strong enough such a geometric mesh is not effective because the major error comes out of the larger internal element. This behavior can be seen in Fig. 4.8a ($\alpha = 80°$, weak singularity). However, after a certain number of successive refinements analogous situation will occur for the other angles as well. When this phase is reached simultaneous refinement around the singularity location and outside should be made.

Figures 4.9a-d show the error in the displacement at the center of the plate. Again, the slope of the theoretical convergence rate (that is twice the order of the energy norm) is indicated.

We see that the ARGY element, which is of higher degree, performs very well in the case of non-uniform mesh. This is among others due to the fact that higher order finite elements are able to better "absorb" the singularities of the solution, as shown in [32], [33] in relation to the p-version of the finite element method.

It is necessary to note that ARGY element is overconstrained (i.e. the second derivatives at the vertices are used). This can sometimes lead to difficulties (not present in the problem we are dealing with). For example, when the plate change thickness the second derivatives of the solution are not continuous while the overconstraining enforces continuity. Nevertheless, a simple adaptation of the elements, using the fact that the character of the discontinuity is known, can avoid the difficulty. Another well known cases are those when boundary condition changes from clamped type to nonclamped and in corners of clamped boundary (which can be delt with by a proper refinement

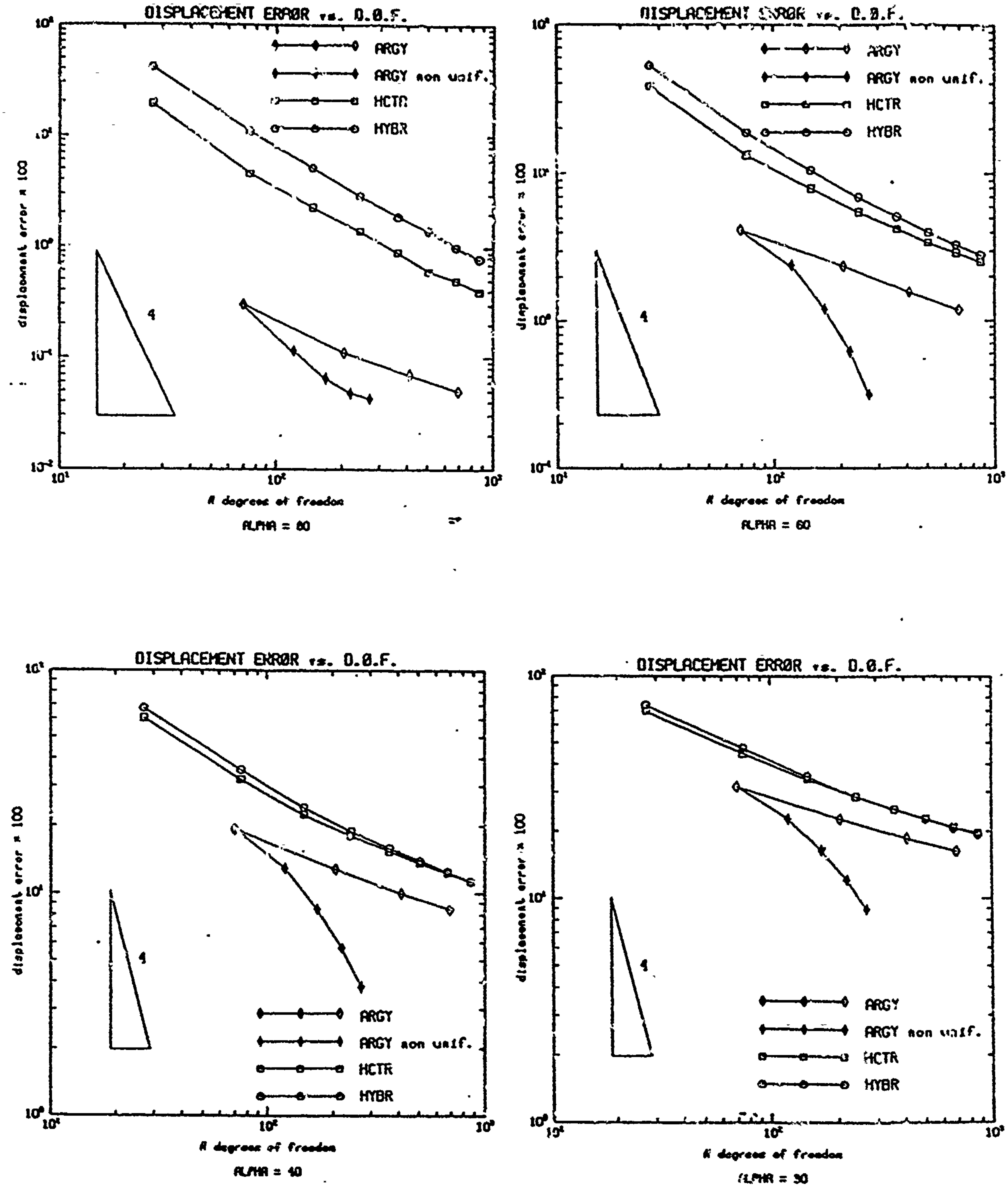

Fig. 4.9a-d. Relative displacement error at the center of the plate against number of degrees of freedom for uniform and non-uniform mesh.

of the mesh). Hence the results of the discussed benchmark is representation for simply supported polygonal plates.

4.3. Accuracy and computational work

The comparison accuracy - number of degrees of freedom is the most used in the scientific literature. However, given a problem to solve within a certain range of precision, the most important relation is between accuracy and cost of computation. This should be, and indeed it is in practical environment, the criterium upon which to rest the selection of the most suitable finite element. Only recently this problem has been approached from a quite general standpoint (see [34]). Hereafter we will give only some general lines of this important question, focusing mainly on the numerical results obtained with the finite elements previously described. The finite element computation requires the execution of the following steps:

a - topology, mesh generation

b - local stiffness matrices

c - assembly and solution

d - postprocessing (i.e. computation of required data).

Sometimes assembly is combined with computation of local stiffness matrices. The cost of the mesh generation and the cost of postprocessing are not too much sensitive to the elements. On the other hand, parts b and c depend heavily on the type of finite element. Toward a simple approach we will only consider the cost of parts b and c. As regards the accuracy, we have already seen that different measures can be used (e.g. energy norm, L_∞ norm, etc.). We will especially consider the energy norm although pointwise accuracy could be examined as well. The conclusion we will base on the energy norm will essentially hold for pointwise accuracy.

Computer time depends on both hardware (operative system, compiler, etc.) and software (programmation technique, etc.). Nevertheless, it is reasonable to assume that the state of the art programming will lead to acceptable and objective conclusions. Of course, some very special techniques could influence the conclusions but this has to be considered as a "different" method (as it is the case for parallel computer oriented methods, which are not optimal for sequential machines and could give better performances).

The results presented throughout the paper has been obtained using an Apollo 410 computer. The programs are performed in a standard way and are included in the finite element library MODULEF (see [35]). The final linear system is solved via elimination method based on the Choleski factorization.

In Fig. 4.10a,b the time for the computation of local stiffness matrices, resp. for assembly and solution, against the number of degrees of freedom is shown. As expected, in both the figures the time spent is larger for ARGY than HCTR and HYBR element, depending mainly on the dimension of the elementary stiffness matrix.

As previously said we are interested in accuracy versus cost and this is the subject of Fig. 4.11a-d where the cost is represented as total time for computation of local stiffness matrices, assembly and solution while the error in tne energy norm is chosen for the accuracy.

Analogous result hold for the displacement error at the center of the plate. We compared the performance of the elements ARGY, HCTR and HYBR on a uniform mesh. In addition, we have also shown the performance of the ARGY element for refined mesh. (Such a mesh needs more work to generate.) The proper mesh refinement has larger effects on the performance of the higher degree element than the lower degree, hence, for the refined mesh the comparison will be still more favorable for ARGY.

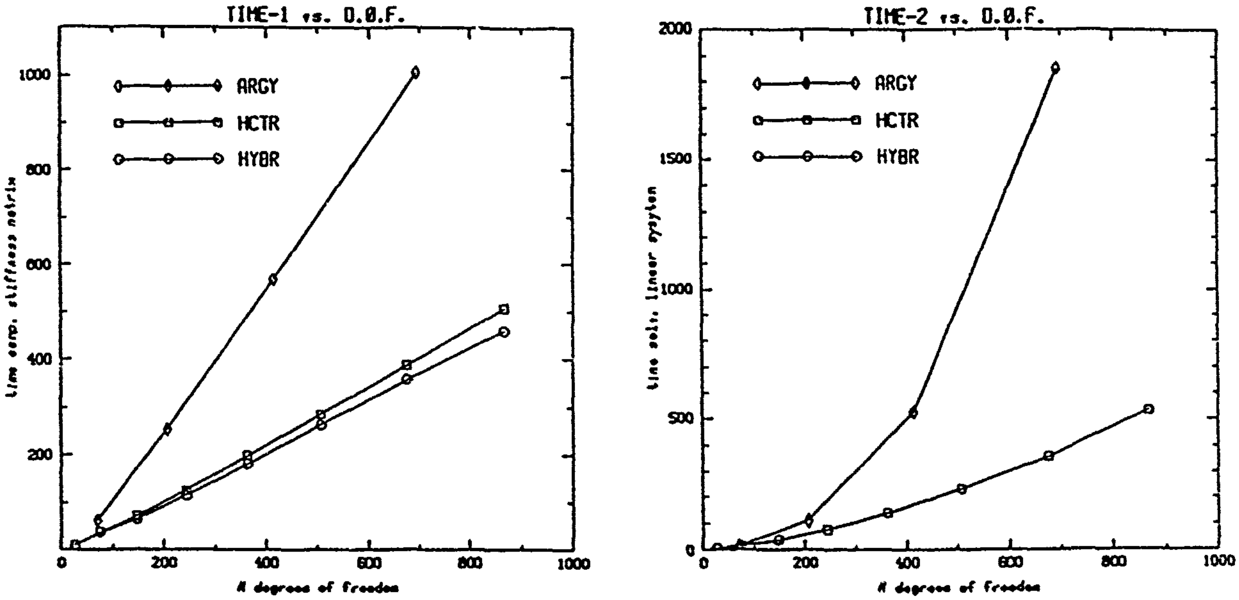

Fig. 4.10a,b. Time for computation of local stiffness matrices against number of degrees of freedom (a). Time for assembly and solution against number of degrees of freedom (B) (HCTR and HYBR connect).

<u>Remark 4.1</u>. We analyzed the performance of the finite element method to solve the plate problem according to the mathematical model based upon the Kirchhoff theory. Of course, the question about the validity of such a model is essential. We can formulate the plate problem as a 3-dimensional elasticity problem and give it (exact) solution the meaning of true solution. Then we can compare this value with the (exact) solution of the Kirchhoff (2-dimensional) model.

It is well known that as the thickness $t \to 0$ the 3-dimensional solution converges to the 2-dimensional one (see, e.g. [36], [37], [38], [40], [41]). The accuracy of the Kirchhoff solution depends very much on both thickness t and skew angle α.

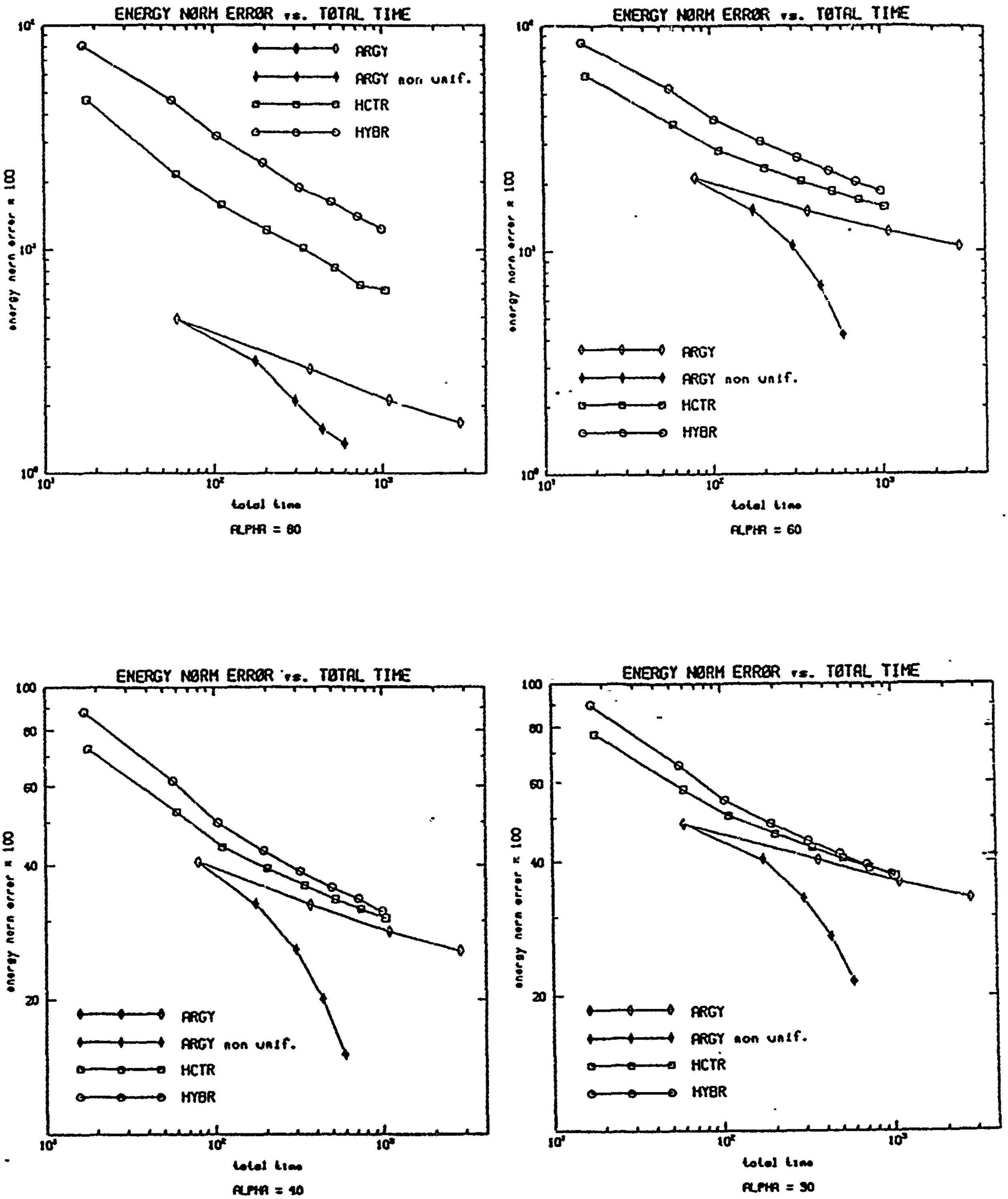

Fig. 4.11a-d. Energy norm of the error against total time for computation (stiffness matrices, assembly, solution).

Denote by u, v, w the 3D displacement of the plate and let us model the simple support by constraining $w = 0$ at the sides boundary surface while leaving u and v unconstrained. Let E_{3D} be the exact (strain) energy of the 3D formulation, E_K the Kirchhoff one. Then the quantity

$$EE = \left(\frac{E_{3D} - E_K}{E_{3D}}\right)^{1/2} \tag{4.4}$$

is very close to the relative error of the Kirchhoff solution in the energy norm (like in relation 4.1). In fact, for $\nu = 0$, ψ is the exact relative error measured in the energy norm.

Since the uniform load is $g = 1$, we have:

$$E_{3D} = \int_\Omega w_{3D}\, dx_1\, dx_2, \tag{4.5}$$

$$E_K = \int_\Omega w_K\, dx_1\, dx_2, \tag{4.6}$$

$$E_{3D} - E_K = \int_\Omega (w_{3D} - w_K) dx_1\, dx_2, \tag{4.7}$$

where w_{3D}, w_K denote the vertical displacement in the 3D and Kirchhoff model, resp. Hence, $ED = (EE)^2$ is the relative average error in the displacement at the loaded surface.

Table 4.1 shows the value EE, ED for $t = .0.01$ and various values α. The values E_{3D} have been computed out of the results obtained using the code STRIPE (with p-version capability), developed by the Computational Mechanics Center of the Aeronautical Research Institute of Sweden (see [39]), on the CRAY X-MP/48 of the Pittsburgh Supercomputing Center (PSC). A careful extrapolation technique, based on the results for different values of p and meshes, has been applied. The values E_K have been computed with analogous extrapolation technique, based on the results for different values of p and meshes, has been applied. The values E_K have been computed with analogous procedures out of the numerical results described in the subsections 4.1-4.3.

TABLE 4.1

SKEW ANGLE	ERR1	ERR2
80	11.65	1.36
60	17.89	3.20
40	33.71	11.37
30	38.17	14.57

ERR1: % ERROR IN THE ENERGY NORM

ERR2: % AVERAGE DISPLACEMENT ERROR

thickness = 0.01
Poisson's ratio = 0.3

We see that the Kirchhoff model gives unacceptable results except for plates either close to the square shape or with very small thickness. In a forthcoming paper we will address in details the reliability of various plate models of Reissner-Mindlin type and discuss the accuracy of their finite element approximation.

5. CONCLUSIONS

Let us summarize the conclusions which can be drawn from the benchmark computation of the uniformly loaded simply supported skew plate with Kirchhoff formulation.

1. The major reason of the error dependance on the skewness of the plate is the singularity of the solution and not the skewness of the element of the decomposition. Hence this benchmark problem is not well suited for comparing different elements except for the case when the performance in presence of strong singularity of the solution needs to be evaluated.

2. The problem characterizes well any polygonal simply supported plate subject to analytic load. The solution in the neighborhood of a critical vertex with internal angle β (see Fig. 5.1) has, for any β such that $\left(\frac{\pi}{\beta}\right)^{-1}$ is not integer, the following expression:

$$u = C_1 r^{2+\frac{\pi K_1}{\beta}} \sin \frac{\pi K_1}{\beta}\theta + C_2 r^{\frac{\pi K_2}{\beta}} \sin \frac{\pi K_2}{\beta}\theta + \text{smoother terms}$$

with K_1, K_2 integers and since $u \in H^2(\Omega)$ we have to have $2 + \frac{\pi K_1}{\beta} > 1$ and $\frac{\pi K_2}{\beta} > 1$. Hence the major singularity term is:

$$u = Cr^{\frac{\pi}{\beta}} \sin \frac{\pi}{\beta}\theta \qquad \text{for} \quad 0 < \beta < \pi$$

$$u = Cr^{2-\frac{\pi}{\beta}} \sin \frac{\pi}{\beta}\theta \qquad \text{for} \quad \pi < \beta < 2\pi.$$

For example, the L-shaped domain with $\beta = \frac{3}{2}\pi$ should be compared with a skew plate with skewness angle $\alpha = 45°$.

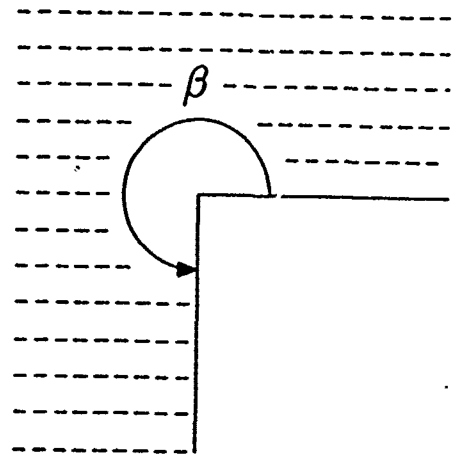

Fig. 5.1. A reentrant corner in a L-shaped domain.

Therefore, with proper correspondence the benchmark problem can be applied quite generally.

3. The higher order elements provide better accuracy (in the engineering range) than the lower ones for the same computational cost, both in presence of singularity and for smooth solution, either for uniform or properly designed meshes. By accuracy we mean the error in the energy norm or pointwise error. The computer cost takes into account computation of local stiffness matrices, assembling procedure and (direct) solving time. The relation between accuracy and cost are in agreement with the theoretical model introduced in [34].

4. Because anisotropic plates, after proper coordinates transformation, has the same (or very similar) properties of isotropic ones, the conclusions hold also for this case. Of course, the notion of the angle is now not only geometrical but depends also on the anisotropy. For example, the finite ele-

ment of an anisotropic square plate behaves then as the solution of an isotropic skew plate.

5. Accuracy of the Kirchhoff model compared with 3 dimensional linear elasticity solution could not be acceptable for larger skewness of the plate.

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